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Analytical Solution of a System of Ordinary Differential Equations

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Abstract

In this paper, we apply the Adomian decomposition method (ADM) for solving linear and nonlinear system of ordinary differential equations (ODEs). The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed. Some numerical examples are solved.

Keywords: *Adomian Method; existence; uniqueness; error analysis.*

1 Introduction

Differential equations have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [1]-[11]. In this paper, Adomian decomposition method (ADM) [12]-[19] is used to solve these type of equations. This method has many advantages, it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization. The convergence of the series solution and the error analysis are discussed. Some numerical examples are solved.

2 Problem solving

2.1 The solution algorithm

Let us consider the system of nonlinear ODEs,

$$\begin{aligned} \mathrm{L}y_i(t) + f_i(y) &= x_i(t), \qquad i = 1, 2, \ldots, m. \\ y_i^{(j)} &= 0, \qquad j = 0, 1, 2, \ldots, n-1. \end{aligned} \tag{1}$$

Where

$$\mathrm{L} = L + R, \tag{2}$$

$$L = \frac{d^n}{dt^n}, \quad \text{and} \quad R = \sum_{k=0}^{n-1} a_k(t)\frac{d^k}{dt^k}. \tag{3}$$

And $f_i(y)$ are the nonlinear terms expanded in terms of Adomian polynomials,

$$f_i(y) = \sum_{n=0}^{\infty} A_{i,n}, \tag{4}$$

$$A_{i,n} = \left(\frac{1}{n!}\right)\frac{d^n}{d\lambda^n}\left[f_i\left(\sum_{j=0}^{\infty}\lambda^j y_j\right)\right]_{\lambda=0}, \tag{5}$$

And the linear operator L as defined before in equations (2) and (3). Substitute from (3) and (2) into (1) we get,

$$(L+R)y_i(t) + \sum_{n=0}^{\infty} A_{i,n} = x_i(t), \tag{6}$$

$$Ly_i(t) = x_i(t) - Ry_i(t) - \sum_{n=0}^{\infty} A_{i,n}, \tag{7}$$

Applying L^{-1} to both sides of equation (7) we have,

$$y_i(t) = L^{-1}x_i(t) - L^{-1}Ry_i(t) - L^{-1}\left(\sum_{n=0}^{\infty} A_{i,n}\right), \tag{8}$$

Decompose $y_i(t) = \sum_{n=0}^{\infty} y_{i,n}(t)$ and substitute in equation (8), we get the following recursive relations,

$$y_{i,0}(t) = L^{-1}x_i(t), \tag{9}$$

$$y_{i,n}(t) = -L^{-1}Ry_{i,n-1}(t) - L^{-1}A_{i,n-1}. \tag{10}$$

Finally, the solution of (1) is

$$y_i(t) = \sum_{n=0}^{\infty} y_{i,n}(t). \tag{11}$$

3 Convergence analysis

3.1. Existence and uniqueness of the solution

Define the mapping $F : E \to E$ where E is the Banach space ($C[I], \|\cdot\|$), the space of all continuous functions on I with the norm $\|f(t)\| = \max_{t \in I} |f(t)|$, $\left|\sum_{k=0}^{n-1} a_k(t)\right| \prec M$ $\forall\, 0 \le \tau \le t \le T$, M is finite constant and $f_i(y)$ satisfy Lipschitz condition with Lipschitz constants C_i such as, constants C_i such as,

$$|f_i(y) - f_i(z)| \le C_i |y_i - z_i|$$

<u>**Theorem 1**</u>:

The problem (1) has a unique solution whenever $0 < \beta < 1$ *where,* $\beta = T^n[M + C]$.

<u>**Proof:**</u>

The mapping $F : E \to E$ is defined as,

$$Fy_i(t) = L^{-1} x_i(t) - L^{-1} Ry_i(t) - L^{-1}(f_i(y_i))$$

Let $y(t), z(t) \in E$:

$$\begin{aligned}
\|Fy_i - Fz_i\| &= \max_{t \in I} \left| -L^{-1} Ry_i(t) - L^{-1}(f(y_i)) + L^{-1} Rz_i(t) + L^{-1}(f_i(z_i)) d\tau \right| \\
&= \max_{t \in I} |[L^{-1} Ry_i(t) - L^{-1} Rz_i(t)] + [L^{-1}(f_i(y_i)) - L^{-1}(f_i(z_i))] d\tau| \\
&\le \max_{t \in I} \left| L^{-1} R[y_i(t) - z_i(t)] \right| + \max_{t \in I} |L^{-1}[f_i(y_i) - f_i(z_i)]| d\tau \\
&\le \max_{t \in I} |y_i(t) - z_i(t)| \left| L^{-1} R[1] \right| + C_i \max_{t \in I} |y_i(t) - z_i(t)| |L^{-1}[1]| d\tau \\
&\le \max_{t \in I} |y_i(t) - z_i(t)| \left| \sum_{k=0}^{n-1} a_k(t) \right| T^{n-k} + C_i T^n \max_{t \in I} |y_i(t) - z_i(t)| \\
&\le [MT^n + C_i T^n] \|y_i - z_i\| \\
&\le T^n[M + C_i] \|y_i - z_i\|
\end{aligned}$$

And let $|C_i| \le C,$ then

$$\begin{aligned}
\|Fy_i - Fz_i\| &\le T^n[M + C] \|y_i - z_i\| \\
&\le \beta \|y_i - z_i\|
\end{aligned}$$

Under the condition, $0 < \beta < 1,$ the mapping F is contraction and hence there exists a unique solution of the problem *(1)-(2)* and this completes the proof. ■

3.2. Proof of convergence

Theorem 2:

The series solution (10) of the problem (1)-(2) using ADM converges if $|y_{j,1}| < \infty$ *and* $0 < \beta < 1,\ \beta = T^n[M + L]$.

Proof: Define the sequence $\{S_{j,n}\}$ such that, $S_{j,n} = \sum_{i=0}^{n} y_{j,i}(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} y_{j,i}(t)$ since,

$$f_j(y) = f_j\left(\sum_{i=0}^{\infty} y_{j,i}(t)\right) = \sum_{i=0}^{\infty} A_{j,i}(y_0, y_1, \ldots, y_i)$$

So,

$$f(S_{j,n}) = \sum_{i=0}^{n} A_{j,i}(y_0, y_1, \ldots, y_i)$$

Let $S_{j,n}$ and $S_{j,m}$ be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_{j,n}\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\| S_{j,n} - S_{j,m} \| &= \max_{t\in I} |S_{j,n} - S_{j,m}| = \max_{t\in I} \left| \sum_{i=m+1}^{n} y_{j,i}(t) \right| \\
&= \max_{t\in I} \left| \sum_{i=m+1}^{n} [L^{-1} R y_{j,i}(t) + L^{-1}(f_j(y_j))] \right| \\
&= \max_{t\in I} \left| \left[L^{-1} R \sum_{i=m+1}^{n} y_{j,i}(t) + L^{-1} \sum_{i=m+1}^{n} A_{j,i} d\tau \right] \right| \\
&= \max_{t\in I} \left| \left[L^{-1} R \sum_{i=m}^{n-1} y_{j,i}(t) + L^{-1} \sum_{i=m}^{n-1} A_{j,i} d\tau \right] \right|
\end{aligned}$$

$$
\begin{aligned}
&= \max_{t\in I} |[L^{-1} R[S_{j,n-1} - S_{j,m-1}] + L^{-1}[f(S_{j,n-1}) - f(S_{j,m-1})]d\tau]| \\
&\leq \max_{t\in I} L^{-1} R|S_{j.n-1} - S_{j,m-1}| + L^{-1}|f(S_{j,n-1}) - f(S_{j,m-1})|d\tau \\
&\leq \max_{t\in I} L^{-1} R|S_{j,n-1} - S_{j,m-1}| + L^{-1}|f(S_{j,n-1}) - f(S_{j,m-1})|d\tau \\
&\leq [MT^n + C_i T^n] \| S_{j,n-1} - S_{j,m-1} \| \\
&\leq \beta \| S_{j,n-1} - S_{j,m-1} \|
\end{aligned}
$$

Let $n = m + 1$ then,

$$
\| S_{j,m+1} - S_{j,m} \| \leq \beta \| S_{j,m} - S_{j,m-1} \| \leq \beta^2 \| S_{j,m-1} - S_{j,m-2} \| \leq \cdots \leq \beta^m \| S_{j,1} - S_0 \|
$$

From the triangle inequality we have,

$$
\begin{aligned}
\| S_{i,n} - S_{i,m} \| &\leq \| S_{j,m+1} - S_{j,m} \| + \| S_{j,m+2} - S_{j,m+1} \| + \cdots + \| S_{j,n} - S_{j,n-1} \| \\
&\leq [\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}] \| S_{j,1} - S_{j,0} \| \\
&\leq \beta^m [1 + \beta + \cdots + \beta^{n-m-1}] \| S_{j,1} - S_{j,0} \| \\
&\leq \beta^m \left[\frac{1 - \beta^{n-m}}{1 - \beta} \right] \| y_{j,1}(t) \|
\end{aligned}
$$

Since, $0 < \beta < 1$, and $n > m$ then, $(1 - \beta^{n-m}) \leq 1$. Consequently,

$$
\begin{aligned}
\| S_{j,n} - S_{j,m} \| &\leq \frac{\beta^m}{1 - \beta} \| y_{j,1}(t) \| \\
&\leq \frac{\beta^m}{1 - \beta} \max_{t\in I} |y_{j,1}(t)|
\end{aligned}
$$

but, $|y_{j,1}(t)| < \infty$ and as $m \to \infty$ then, $\| S_{j,n} - S_{j,m} \| \to 0$ and hence, $\{S_{j,n}\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{n=0}^{\infty} y_{j,n}(t)$ converges and this completes the proof. ■

3.3. Error analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

Theorem 3: *The maximum absolute truncation error of the series solution (10) to the problem (1)-(2) is estimated to be,*

$$\max_{t\in J}\left|y_j(t)-\sum_{i=0}^{m}y_{j,i}(t)\right|\le\frac{\beta^m}{1-\beta}\max_{t\in I}|y_{j,1}(t)|$$

Proof: From Theorem 2 we have,

$$\|S_{j,n}-S_{j,m}\|\le\frac{\beta^m}{1-\beta}\max_{t\in I}|y_{j,1}(t)|$$

But, $S_{j,n}=\sum_{i=0}^{n}y_{j,i}(t)$ as $n\to\infty$ then, $S_{j,n}\to y_j(t)$ so,

$$\|y_j(t)-S_{j,m}\|\le\frac{\beta^m}{1-\beta}\max_{t\in I}|y_{j,1}(t)|$$

Therefore, the maximum absolute truncation error in the interval I is,

$$\max_{t\in I}\left|y_j(t)-\sum_{i=0}^{m}y_{j,i}(t)\right|\le\frac{\beta^m}{1-\beta}\max_{t\in I}|y_{j,1}(t)|$$

And this completes the proof. ■

4. Numerical Examples

Example 1: Consider the following linear system of ODEs,

$$\begin{aligned}\frac{dy_1}{dt}&=2t-t^3+y_2,\\ \frac{dy_2}{dt}&=3y_1,\\ \frac{dy_3}{dt}&=4y_2,\end{aligned}\qquad(12)$$

Subject to the initial conditions,

$$y_1(0)=y_2(0)=y_3(0)=0,$$

Which has the exact solution $y_1(t)=t^2$, $y_2(t)=t^3$ ***and*** $y_3(t)=t^4$.

Applying ADM to the system (12) we have,

$$y_{1,0}=t^2-\frac{t^4}{4},\quad y_{1,j+1}=\int_0^t y_{2,j}(\tau)d\tau,\qquad(13)$$

$$y_{2,0} = 0, \quad y_{2,j+1} = 3\int_0^t y_{1,j}(\tau)\,d\tau, \tag{14}$$

$$y_{3,0} = 0, \quad y_{3,j+1} = 4\int_0^t y_{2,j}(\tau)\,d\tau. \tag{15}$$

Using the relations (13)-(15), the first three-terms of the series solution are,

$$y_1 = t^2 - \frac{t^4}{4} - \frac{1}{40}t^4(-10 + t^2) + \cdots, \tag{16}$$

$$y_2 = t^3 - \frac{3t^5}{20} + \cdots, \tag{17}$$

$$y_3 = t^4 - \frac{t^6}{10} + \cdots. \tag{18}$$

Figures 1.a-1.c show a comparison between the exact and ADM solutions of y_1, y_2 and y_3

($n = 50$ **).**

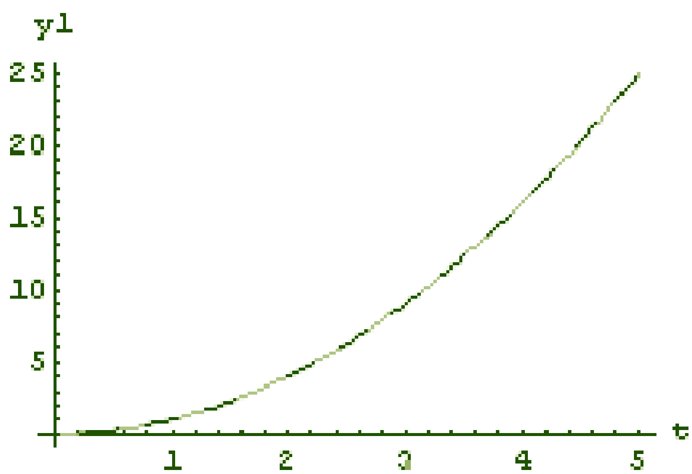

Figure 1.a: ADM and Exact Sol. y1

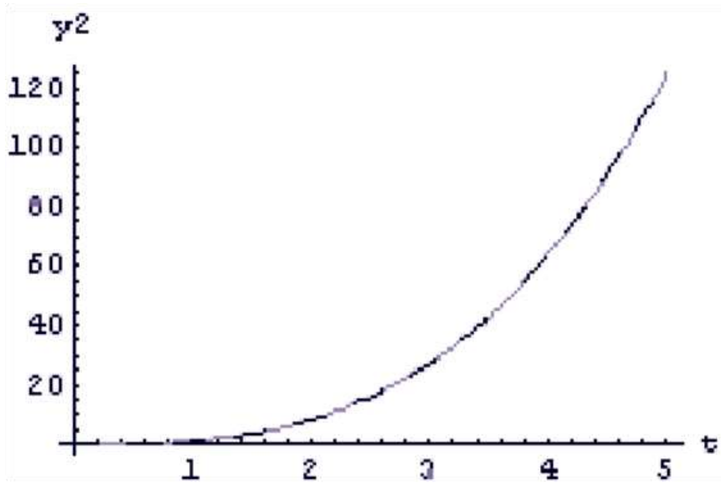

Figure 1.b: ADM and Exact Sol. y2

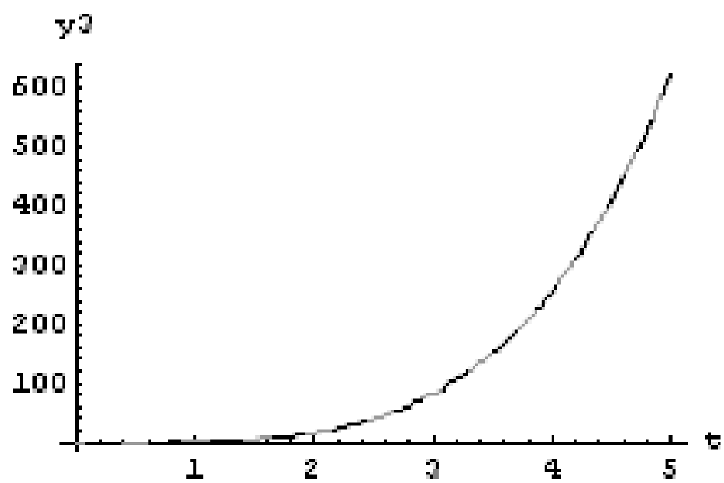

Figure 2.c: ADM and Exact Sol. y3

Example 2: Consider the following nonlinear system of ODEs,

$$\frac{d^2 y_1}{dt^2} = 2 - t^9 + y_2^3,$$
$$\frac{d^2 y_2}{dt^2} = 6t - t^8 + y_1^4, \tag{19}$$

Subject to the initial conditions,

$$y_1(0) = y_1'(0) = y_2(0) = y_2'(0) = 0,$$

Which has the exact solution $y_1(t) = t^2$ ***and*** $y_2(t) = t^3$.

Applying ADM to the system (19) we have,

$$y_{1,0} = t^2 - \frac{t^{11}}{110}, \quad y_{1,j+1} = \int_0^t \int_0^t A_{1,j}(\tau)\,d\tau\,d\tau,$$
$$y_{2,0} = t^3 - \frac{t^{10}}{90}, \quad y_{2,j+1} = \int_0^t \int_0^t A_{2,j}(\tau)\,d\tau\,d\tau. \tag{20}$$

Using the relations (20), the first two-terms of the series solution are,

$$y_1 = t^2 - \frac{t^{18}}{9180} + \frac{t^{25}}{162000} - \frac{t^{32}}{723168000} + \cdots, \tag{21}$$

$$y_2 = t^3 - \frac{t^{19}}{9405} + \frac{t^{28}}{1524600} - \frac{t^{37}}{443223000} + \frac{t^{46}}{303068700000} + \cdots. \tag{22}$$

Figures 2.a and 2.b show ADM solution of y_1, y_2 ($n = 5$) .

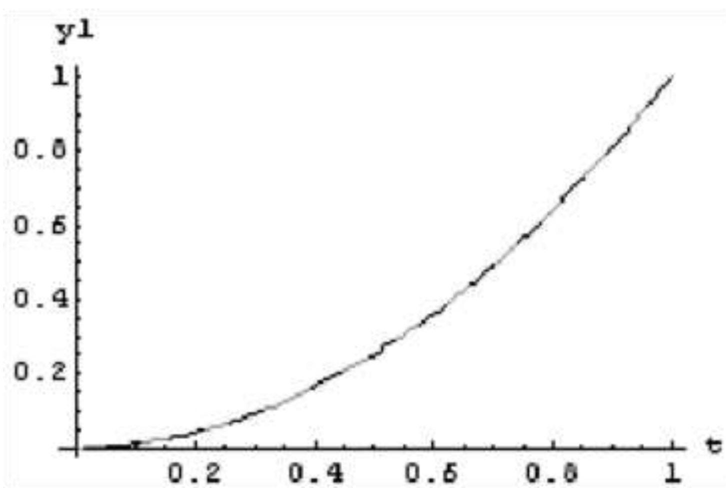

Figure 2.a: ADM and Exact Sol. y1

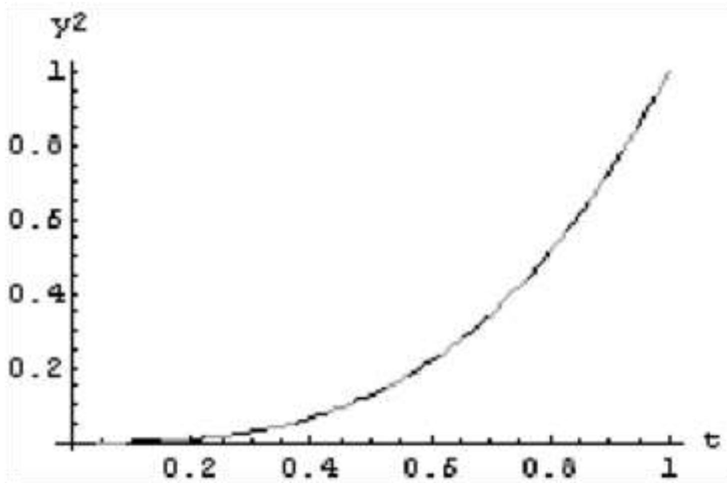

Figure 2.b: ADM and Exact Sol. y2

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