Analytical Solution of a System of Ordinary Differential Equations

E. A. A. Ziada
Nile Higher Institute for Engineering and Technology, Mansoura, Egypt.
eng_emanziada@yahoo.com

Abstract
In this paper, we apply the Adomian decomposition method (ADM) for solving linear and nonlinear system of ordinary differential equations (ODEs). The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed. Some numerical examples are solved.

Keywords: Adomian Method; existence; uniqueness; error analysis.

1 Introduction
Differential equations have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [1]-[11]. In this paper, Adomian decomposition method (ADM) [12]-[19] is used to solve these type of equations. This method has many advantages, it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization. The convergence of the series solution and the error analysis are discussed. Some numerical examples are solved.

2 Problem solving
2.1 The solution algorithm
Let us consider the system of nonlinear ODEs,

\[ L y_i(t) + f_i(y) = x_i(t), \quad i = 1, 2, \ldots, m \]
\[ y_i^{(j)} = 0, \quad j = 0, 1, 2, \ldots, n - 1. \]  

(1)
Where
\[ L = L + R, \]  \hspace{1cm} (2)

\[ L = \frac{d^n}{dt^n}, \quad \text{and} \quad R = \sum_{k=0}^{n-1} a_k(t) \frac{d^k}{dt^k}. \]  \hspace{1cm} (3)

And \( f_i(y) \) are the nonlinear terms expanded in terms of Adomian polynomials,

\[ f_i(y) = \sum_{n=0}^{\infty} A_{i,n}, \]  \hspace{1cm} (4)

\[ A_{i,n} = \left( \frac{1}{n!} \right) \frac{d^n}{d\lambda^n} \left[ f_i \left( \sum_{j=0}^{\infty} \lambda^j y_j \right) \right]_{\lambda=0}. \]  \hspace{1cm} (5)

And the linear operator \( L \) as defined before in equations (2) and (3). Substitute from (3) and (2) into (1) we get,

\[ (L + R)y_i(t) + \sum_{n=0}^{\infty} A_{i,n} = x_i(t), \]  \hspace{1cm} (6)

\[ Ly_i(t) = x_i(t) - Ry_i(t) - \sum_{n=0}^{\infty} A_{i,n}, \]  \hspace{1cm} (7)

Applying \( L^{-1} \) to both sides of equation (7) we have,

\[ y_i(t) = L^{-1} x_i(t) - L^{-1} Ry_i(t) - L^{-1} \left( \sum_{n=0}^{\infty} A_{i,n} \right); \]  \hspace{1cm} (8)

\[ y_i(t) = \sum_{n=0}^{\infty} y_{i,n}(t) \]

Decompose \( y_i(t) \) and substitute in equation (8), we get the following recursive relations,

\[ y_{i,0}(t) = L^{-1} x_i(t), \]  \hspace{1cm} (9)

\[ y_{i,n}(t) = -L^{-1} R y_{i,n-1}(t) - L^{-1} A_{i,n-1}. \]  \hspace{1cm} (10)

Finally, the solution of (1) is

\[ y_i(t) = \sum_{n=0}^{\infty} y_{i,n}(t). \]  \hspace{1cm} (11)

### 3 Convergence analysis

#### 3.1. Existence and uniqueness of the solution
Define the mapping $F: E \to E$ where $E$ is the Banach space $(C[I], \| \cdot \|)$, the space of all continuous functions on $I$ with the norm $\| \mathcal{A}(\theta) \| = \max_{t \in I} |\mathcal{A}(\theta)|$, $\sum_{k=0}^{n-1} a_k(\theta) < M$.

where $E$ is the Banach space $(C[I] \to \mathbb{R}, \| \cdot \|)$, the space of all continuous functions on $I$ with the norm $\| f \|_{\max} = \max_{t \in I} |f(t)|$.

And let $|C_i| \leq C$, then

$$\| F y_i - F z_i \| \leq T^n [M + C] \| y_i - z_i \|$$

$$\leq \beta \| y_i - z_i \|$$

Under the condition, $0 < \beta < 1$, the mapping $F$ is contraction and hence there exists a unique
solution of the problem (1)-(2) and this completes the proof. ■

3.2. Proof of convergence

Theorem 2:

The series solution (10) of the problem (1)-(2) using ADM converges if \(|y_{j,1}| < \infty\) and \(0 < \beta < 1\), \(\beta = T^n[M+L]\).

Proof: Define the sequence \(\{S_{j,n}\}\) such that, \(S_{j,n} = \sum_{i=0}^{n} y_{j,i}(\hat{t})\) is the sequence of partial sums from the series solution \(\sum_{i=0}^{\infty} y_{j,i}(\hat{t})\) since,

\[
f_j(y) = f_j\left(\sum_{i=0}^{\infty} y_{j,i}(\hat{t})\right) = \sum_{i=0}^{\infty} A_{j,i}(y_0, y_1, \ldots, y_i)
\]

So,

\[
\mathcal{S}(S_{j,n}) = \sum_{i=0}^{n} A_{j,i}(y_0, y_1, \ldots, y_i)
\]

Let \(S_{j,n}\) and \(S_{j,m}\) be two arbitrary partial sums with \(n > m\). Now, we are going to prove that \(\{S_{j,n}\}\) is a Cauchy sequence in this Banach space.

\[
\|S_{j,n} - S_{j,m}\| = \max_{t \in I} |S_{j,n} - S_{j,m}| = \max_{t \in I} \left| \sum_{i=m+1}^{n} y_{j,i}(\hat{t}) \right|
\]

\[
= \max_{t \in I} \left| \sum_{i=m+1}^{n} [L^{-1} R y_{j,i}(\hat{t}) + L^{-1} f_j(y)] \right|
\]

\[
= \max_{t \in I} \left| L^{-1} R \sum_{i=m+1}^{n} y_{j,i}(\hat{t}) + L^{-1} \sum_{i=m+1}^{n} A_{j,i} \right|
\]

\[
= \max_{t \in I} \left| L^{-1} R \sum_{i=m}^{n} y_{j,i}(\hat{t}) + L^{-1} \sum_{i=m}^{n-1} A_{j,i} \right|
\]
\[
= \max_{t \in I} \left[ L^{-1} R[S_{j,n-1} - S_{j,m-1}] + L^{-1} [\kappa(S_{j,n-1}) - \kappa(S_{j,m-1})] dt \right] \\
\leq \max_{t \in I} L^{-1} R|S_{j,n-1} - S_{j,m-1}| + L^{-1} |\kappa(S_{j,n-1}) - \kappa(S_{j,m-1})| dt \\
\leq \max_{t \in I} L^{-1} R|S_{j,n-1} - S_{j,m-1}| + L^{-1} |\kappa(S_{j,n-1}) - \kappa(S_{j,m-1})| dt \\
\leq [MT^n + C_i T^n]\|S_{j,n-1} - S_{j,m-1}\| \\
\leq \beta \|S_{j,n-1} - S_{j,m-1}\|
\]

Let \( n = m+1 \) then,
\[
\|S_{j,m+1} - S_{j,m}\| \leq \beta \|S_{j,m} - S_{j,m-1}\| \leq \beta^2 \|S_{j,m-1} - S_{j,m-2}\| \leq \cdots \leq \beta^m \|S_{j,1} - S_0\|
\]

From the triangle inequality we have,
\[
\|S_{i,n} - S_{i,m}\| \leq \|S_{i,n+1} - S_{i,m}\| + \|S_{i,m+2} - S_{i,m+1}\| + \cdots + \|S_{i,n} - S_{i,n-1}\| \\
\leq [\beta^m + \beta^m + \cdots + \beta^m] \|S_{i,1} - S_{i,0}\| \\
\leq \beta^m [1 + \beta + \cdots + \beta^{m-1}] \|S_{i,1} - S_{i,0}\| \\
\leq \beta^m \left[ \frac{1 - \beta^{m-1}}{1 - \beta} \right] \|y_{i,1}(t)\|
\]

Since, \( 0 < \beta < 1 \), and \( n > m \) then, \( (1 - \beta^{n-m}) \leq 1 \). Consequently,
\[
\|S_{j,n} - S_{j,m}\| \leq \frac{\beta^m}{1 - \beta} \|y_{j,1}(t)\| \\
\leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_{j,1}(t)|
\]

but, \( |y_{j,1}(t)| < \infty \) and as \( m \to \infty \) then, \( \|S_{j,n} - S_{j,m}\| \to 0 \) and hence, \( \{S_{j,n}\} \) is a Cauchy sequence in this Banach space so, the series \( \sum_{n=0}^{\infty} y_{j,0}(t) \) converges and this completes the proof.

**3.3. Error analysis**

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

**Theorem 3:** The maximum absolute truncation error of the series solution (10) to the problem (1)-(2) is estimated to be,
\[
\max_{t \in J} \left| y_j(t) - \sum_{i=0}^m y_{j,i}(t) \right| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_{j,1}(t)|
\]

**Proof:** From Theorem 2 we have,

\[
\|S_{j,n} - S_{j,m}\| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_{j,1}(t)|
\]

But, \(S_{j,n} = \sum_{i=0}^n y_{j,i}(t)\) as \(n \to \infty\) then, \(S_{j,n} \to y_j(t)\) so,

\[
\|y_j(t) - S_{j,m}\| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_{j,1}(t)|
\]

Therefore, the maximum absolute truncation error in the interval \(I\) is,

\[
\max_{t \in I} \left| y_j(t) - \sum_{i=0}^m y_{j,i}(t) \right| \leq \frac{\beta^m}{1 - \beta} \max_{t \in I} |y_{j,1}(t)|
\]

And this completes the proof. ■

**4. Numerical Examples**

**Example 1:** Consider the following linear system of ODEs,

\[
\begin{align*}
\frac{dy_1}{dt} &= 2t - t^3 + y_2, \\
\frac{dy_2}{dt} &= 3y_1, \\
\frac{dy_3}{dt} &= 4y_2,
\end{align*}
\]

Subject to the initial conditions,

\[
y_1(0) = y_2(0) = y_3(0) = 0,
\]

Which has the exact solution \(y_1(t) = t^2\), \(y_2(t) = t^3\) and \(y_3(t) = t^4\).

Applying ADM to the system (12) we have,

\[
y_{1,0} = t^2 - \frac{t^4}{4}, \quad y_{1,j+1} = \int_0^t y_{2,j}(\tau) d\tau,
\]

(13)
\[ y_{2,0} = 0, \quad y_{2,j+1} = 3 \int_{0}^{\tau} y_{1,j}(\tau) \, d\tau. \]  \hfill (14)

\[ y_{3,0} = 0, \quad y_{3,j+1} = 4 \int_{0}^{\tau} y_{2,j}(\tau) \, d\tau. \]  \hfill (15)

Using the relations (13)-(15), the first three-terms of the series solution are,

\[ y_1 = \ell^2 - \frac{\ell^4}{4} - \frac{1}{40} \ell^4 (10 + \ell^2) + \cdots, \]  \hfill (16)

\[ y_2 = \ell^3 - \frac{3\ell^5}{20} + \cdots, \]  \hfill (17)

\[ y_3 = \ell^4 - \frac{\ell^6}{10} + \cdots. \]  \hfill (18)

Figures 1.a-1.c show a comparison between the exact and ADM solutions of \( y_1, y_2 \) and \( y_3 \) (\( n = 50 \)).
Example 2: Consider the following nonlinear system of ODEs,

\[
\begin{align*}
\frac{d^2 y_1}{dt^2} &= 2 - t^2 + y_2^3, \\
\frac{d^2 y_2}{dt^2} &= 6t - t^8 + y_1^4,
\end{align*}
\]

(19)
Subject to the initial conditions,

\[ y_1(0) = y'_1(0) = y_2(0) = y'_2(0) = 0, \]

Which has the exact solution \( y_1(t) = t^2 \) and \( y_2(t) = t^3 \).

Applying ADM to the system (19) we have,

\[
\begin{align*}
y_{1,0} & = t^2 - \frac{t^{11}}{110}, & y_{1,j+1} & = \int_0^t \int_0^\tau A_{1,j}(\tau) d\tau d\tau, \\
y_{2,0} & = t^3 - \frac{t^{10}}{90}, & y_{2,j+1} & = \int_0^t \int_0^\tau A_{2,j}(\tau) d\tau d\tau.
\end{align*}
\]

(20)

Using the relations (20), the first two terms of the series solution are,

\[
\begin{align*}
y_1 & = t^2 - \frac{t^{18}}{9180} + \frac{t^{25}}{162000} - \frac{t^{32}}{723168000} + \cdots, \\
y_2 & = t^3 - \frac{t^{19}}{9405} + \frac{t^{28}}{1524600} - \frac{t^{37}}{443223000} + \frac{t^{46}}{303068700000} + \cdots.
\end{align*}
\]

(21) (22)

Figures 2.a and 2.b show ADM solution of \( y_1, y_2 \) ( \( n = 5 \)).
Figure 2.b: ADM and Exact Sol. y2

References

9) Hasanen A. Hammad, and Manuel De la Sen (2021), Tripled fixed point techniques for solving system of tripled-fractional differential equations, AIMS Mathematics 6 (3) 2330-2343.